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Anomalous diffusion in a unidirectional random velocity field with long-range correlations

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Abstract. The anomalous diffusion in a unidirectional random velocity field with long-range correlations is analysed in directions transverse to the direction of the field or along the field direction. Critical values of the exponents, which characterize the power-like falloff of the correlations in the transverse and longitudinal directions, and the critical dimension of space are determined. The anomalous dimension of the longitudinal diffusion coefficient is also calculated in the first order of the ε-expansion for several cases of long-range correlated random velocity field.

1. Introduction

In random media anomalous diffusion is determined by the structure and the range of correlations of the effective random drift field in which the diffusion takes place. Several spatial structures of the random convection field have been considered: it may be divergenceless corresponding to an incompressible solvent, it may be a purely potential vector field or contain both components in different proportions [1, 2], and in all these cases the correlations of the field may have, apart from the tensor structure, either short range (the correlation function is approximated by a delta function) or long range (power-like falloff of the correlation function) [3, 4]. Therefore, there is also a large number of different universality classes characterized by different sets of the critical exponents, which, in particular, determine the power-like behaviour of the mean-square displacement in these models of diffusion. In addition to these features, the drift field may possess, in physically realizable cases, spatial anisotropy [5], which again drastically changes the asymptotic properties of diffusion in such random fields [6, 7].

In this paper long-range generalizations of a recently proposed [5, 6] model of diffusion in a unidirectional random field are considered, with *superdiffusive* asymptotic behaviour, i.e. the mean-square displacement of a tracer particle grows faster than linearly with time. The original model [5] and its generalizations [6, 7] belong to the class of models with divergenceless velocity field, but due to the anisotropic structure of the fields, they are in a different universality class than the models with spatially isotropic divergenceless velocity field [1–3].

The original model is connected with the description of ground water transport in heterogeneous rocks [5]. The basic feature of the unidirectional velocity field in this model is that it is independent of the coordinate along the field direction, whereas it varies in an uncorrelated fashion in the transverse directions. In two dimensions the situation may be described as the diffusion in a fluid flowing along the stripes in a stratified structure with different permeabilities in each stripe. In higher dimensions the system consists of filaments instead of stripes parallel to the flow direction. The properties of diffusion in this unidirectional convection model and its generalizations ('Manhattan grid' convection) have recently been studied both analytically and numerically [6], and also by the method of the renormalization group (RG) [7]. In particular, it has been shown that the critical dimension of these models is $d_c = 3$, and for a random convection field consisting of d' orthogonal unidirectional components, independent of the coordinate along the component direction, the critical exponent ν of the mean-square displacement has been determined as $\nu = (4 + d' - d)/2(d' + 1)$ for an arbitrary dimension of space $d < 3$, and $1 \leqslant d' \leqslant d$.

In this paper, analogous results are obtained for generalizations of the unidirectional model to the case of long-range correlations. Three different possibilities are considered:

(i) the convection field is independent of the coordinate along the field direction, and has long-range correlations in the transverse directions;

(ii) the convection field depends on all the coordinates, but the correlations along the field direction have a long range, whereas in the tranverse direction short-range correlations take place;

(iii) the correlations, both in the field direction and in the transverse directions, have long range.

Physically, (i) corresponds to the situation, in which the permeabilities of the layers or filaments cannot be regarded as totally uncorrelated, e.g. when the characteristic length scale of the changes in the permeability in the transverse directions is larger than the typical width of the filaments or layers; (ii) allows for the coordinate dependence of the random field along its direction. In the filament picture this means that the permeability of each filament varies randomly along the filament direction. However, the randomness induced by the inhomogeneities is assumed to have long-range correlations to account for the larger length scale of the field variations along the field direction, than in the transverse directions, in which the correlations are assumed to be short ranged. In this case, the velocity field ceases to be divergenceless. The last case (iii) is a combination of (i) and (ii).

The paper is organized as follows: in section 2 the field theory of the unidirectionally biased diffusion with long-range transverse correlations is constructed and its renormalization analysed. Generalizations of this model to the cases of long-range longitudinal with short-range transverse correlations and long-range correlations in all directions are discussed in sections 3 and 4, respectively. Section 5 is devoted to concluding remarks.

2. Field theory of diffusion in a random unidirectional velocity field with long-range transverse correlations

Consider a d-dimensional continuum system with stationary random velocity field $\boldsymbol{F}$ in the x-direction: $\boldsymbol{F} = \boldsymbol{e}_x \psi(\boldsymbol{y})$, where the function ψ is a function of the transverse $\boldsymbol{y}$-coordinate only. The motion of a tracer particle at $(x, \boldsymbol{y})$ may be described by the Langevin equations:

$$\mathrm{d}x/\mathrm{d}t = -\psi(\boldsymbol{y}) + \eta_1(t)$$

$$\mathrm{d}y_n/\mathrm{d}t = \eta_n(t) \qquad n > 1 \tag{1}$$

where the Gaussian noise η has zero mean and the correlation functions are

$$\overline{\eta_n(t)\eta_m(t')} = 2\delta_{nm} D_n \delta(t-t') \qquad D_1 = D_0^{\mathrm{L}} \qquad D_n = D_0^{\mathrm{T}} \qquad n > 1 \tag{2}$$

where D_0^{L} is the bare (not renormalized) diffusion coefficient in the x-direction, and D_0^{T} is the bare transverse diffusion coefficient. The random field ψ also has a Gaussian distribution with zero mean, and the correlation function is assumed to be locally integrable in the position space and to have a power-like behaviour at large separations of the field arguments, i.e.

$$\langle\psi(y)\psi(y')\rangle \sim \frac{1}{|y-y'|^{2b}} \qquad |y-y'| \to \infty\,.$$

Therefore, for $b < (d-1)/2$ the Fourier transform of the correlation function behaves as $\langle\psi(p)\psi(q)\rangle \sim \delta(p+q)/(p^2)^{(d-1)/2-b}$ at small momenta, whereas for $b \geqslant (d-1)/2$ we have $\langle\psi(p)\psi(q)\rangle \sim \delta(p+q)$ in the same limit. Therefore, at the large distance limit we may replace the original correlation function by an effective one, which we obtain by taking the inverse Fourier transform of these limiting expressions, and we arrive at the following definition of the effective correlation function

$$\begin{aligned}\langle\psi(y)\psi(y')\rangle &= \frac{2^{2b}\Gamma(b)\lambda_0}{(4\pi)^{(d-1)/2}\Gamma((d-1)/2-b)|y-y'|^{2b}} \\ &\equiv \lambda_0 C_1(y-y') \qquad 0 < b < (d-1)/2 \end{aligned} \tag{3}$$

$$\langle\psi(y)\psi(y')\rangle = \lambda_0\delta(y-y') \equiv \lambda_0 C_1(y-y') \qquad b \geqslant (d-1)/2.$$

Here, Γ is the gamma function, and the (non-negative) bare coupling constant λ_0 describes the strength of the disorder. For convenience, the coefficient of the power of the coordinate difference in the long-range correlation function has been chosen such that $2^{2b}\Gamma(b)/(4\pi)^{(d-1)/2}\Gamma((d-1)/2-b)|y-y'|^{2b} \to \delta(y-y')$, when $b \to (d-1)/2$. The latter case of short-range correlations has already been analysed [7], therefore we concentrate here on the case of long-range correlations.

For the probability distribution $P(t,x,y)$ of the tracer particle at the point (x,y) in a fixed field ψ we obtain the Fokker–Planck equation corresponding to the equations (1) and (2) in the form of the following diffusion equation

$$\left[\frac{\partial}{\partial t} - D_0^{\mathrm{T}}\frac{\partial^2}{\partial y^2} - D_0^{\mathrm{L}}\frac{\partial^2}{\partial x^2} - \psi(y)\frac{\partial}{\partial x}\right] P(t,x,y) \equiv L_\psi P(t,x,y) = 0\,.$$

We are interested in the Green function of this equation, averaged over the random field ψ. This stochastic problem may be cast into a field-theoretic form by the use of the functional-integral representation of the Green function

$$G_\psi(t-t', x-x', y, y') = \int D\varphi D\tilde\varphi\, \varphi(t,x,y)\tilde\varphi(t',x',y') \exp\left[\int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\, \tilde\varphi L_\psi \varphi\right].$$

The Green function averaged over the distribution (3), $\langle G_\psi \rangle$, may be expressed as the Green function $G_0 = \langle G_\psi \rangle$ of the fields φ and $\tilde{\varphi}$ of the field theory with the 'action'

$$S = -\frac{1}{2\lambda_0} \int \mathrm{d}y\, \mathrm{d}y'\, \psi(y) C_1^{-1}(y-y') \psi(y') + \int \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}y\, \tilde{\varphi}(t,x,y) \left[\frac{\partial}{\partial t} - D_0^{\mathrm{T}} \frac{\partial^2}{\partial y^2} - D_0^{\mathrm{L}} \frac{\partial^2}{\partial x^2} - \psi(y) \frac{\partial}{\partial x} \right] \varphi(t,x,y) \tag{4}$$

i.e. as a functional integral over the three fields φ, $\tilde{\varphi}$ and ψ

$$G_0(t-t', x-x', y-y') = \int D\varphi D\tilde{\varphi} D\psi \varphi(t,x,y) \tilde{\varphi}(t',x',y') \exp(S).$$

The graphical rules for the diagrammatic expansion of the Green function G_0 follow from expression (4) in the standard way [8, 9]. It should be noted, however, that the field ψ does not depend on the variables t and x, therefore in the Laplace-momentum representation of the averaged Green function there are no integrals over the Laplace variable and the momentum corresponding to the longitudinal coordinate x. To determine the critical dimension of the field theory (4), we extend the two-scale approach proposed for critical dynamics [10], to the present model with three different scales, and introduce for each variable v three scaling dimensions d_v^{s}, d_v^{L}, and d_v^{T} corresponding to time, longitudinal and transverse coordinates, respectively. The scaling dimensions of all variables are determined from the condition that the action (4) is invariant under scale transformations with respect to time, longitudinal, and transverse coordinates separately. We are interested in a scale transformation, in which the bare propagator g_0 of the field theory (4) in the Laplace-momentum representation

$$g_0(s,k,p) = \frac{1}{s + D_0^{\mathrm{L}} k^2 + D_0^{\mathrm{T}} p^2}$$

is a homogeneous function of order -2 of its Laplace and momentum arguments

$$g_0(\Lambda^2 s, \Lambda k, \Lambda p) = \Lambda^{-2} g_0(s,k,p). \tag{5}$$

The total scaling dimension of a variable v in such a scale transformation is therefore

$$d_v \equiv 2 d_v^{\mathrm{s}} + d_v^{\mathrm{L}} + d_v^{\mathrm{T}}.$$

For example, for the diffusion coefficients we obtain $d_{D^{\mathrm{T}}}^{\mathrm{s}} = d_{D^{\mathrm{L}}}^{\mathrm{s}} = 1$, $d_{D^{\mathrm{T}}}^{\mathrm{L}} = d_{D^{\mathrm{L}}}^{\mathrm{T}} = 0$, and $d_{D^{\mathrm{L}}}^{\mathrm{L}} = d_{D^{\mathrm{T}}}^{\mathrm{T}} = -2$, which yield for the total dimensions the value $d_{D^{\mathrm{L}}} = d_{D^{\mathrm{L}}} = 0$. The scaling dimensions of bare and renormalized diffusion coefficients are the same, therefore we have omitted the subscript '0' in the preceding formulae. This is not so in the case of the coupling constant, therefore the subscript must be retained. For the coupling constant λ_0 we obtain $d_{\lambda_0}^{\mathrm{s}} = 2$, $d_{\lambda_0}^{\mathrm{L}} = -2$, and $d_{\lambda_0}^{\mathrm{T}} = -2b$, therefore $d_{\lambda_0} = 2(1-b)$, from which it follows that the total dimension of the coupling constant vanishes, when $b_{\mathrm{c}} c = 1$. Usually this condition determines the (upper) critical dimension of the model, but we see that there is no critical dimension in the case of algebraically decaying correlations (3), rather a critical value of the exponent b, which characterizes the falloff of the correlations, is determined by the condition $d_{\lambda_0} = 0$, regardless of the space dimension.

Power counting in the graphs shows that the field theory (4) at the critical value of b: $b_c = 1$ is not only renormalizable, but even *super-renormalizable*, i.e. it only possesses a finite number of superficially divergent graphs. To determine whether a model is renormalizable or not, it is customary to calculate the degree of divergence δ of one-particle irreducible (1PI) graphs of the model, and usually δ is equal to the to total dimension defined as the total dimension of the graph in the Laplace-momentum representation. In our case, however, there are no integrals over the Laplace variables and longitudinal momenta in the non-vanishing graphs of the model, therefore the actual degree of divergence δ' is determined by the integrals over transverse momenta only. Power counting of the transverse momenta in an arbitrary 1PI graph yields for the degree of divergence in the transverse momentum space $\delta_T = \delta'$ the expression

$$\delta' = d - 1 - (2 - b)V - (d - 3)N_\varphi - bN_\psi \tag{6}$$

where V is the number of interaction vertices in the graph, and N_φ, N_ψ are the numbers of external φ and ψ legs of the graph, respectively. At the critical value $b = 1$ we obtain $\delta' = d - 1 - V - (d - 3)N_\varphi - N_\psi$, and the coefficient of V does not vanish, although it usually does in a critical theory. The reason is that δ' characterizes large-momentum behaviour of the integrals corresponding to the graphs of the perturbation expansion, but we are eventually interested in the small momentum–small Laplace variable behaviour conforming to the scaling (5). In the scale transformation $s \to \Lambda^2 s$, $k \to \Lambda k$, $p \to \Lambda p$, the original (transverse) large-momentum cutoff Q, which has a fixed value related to the minimal physical length $l \sim 1/Q$ of the problem, is replaced by Q/Λ. For small Λ this may lead to divergences at large momenta in the graphs of the rescaled model, and therefore the large-momentum behaviour of the corresponding integrals has to be analysed. However, in the limit $\Lambda \to 0$, the longitudinal momentum factors at the vertices compensate for the large-momentum divergences of the rescaled model, which has to be taken into account in the investigation of the applicability of the usual perturbation expansion in the small momentum limit. When the scaling dimension of the bare coupling constant d_{λ_0} is positive, the effective (rescaled) coupling constant remains small at small Λ, whereas for $d_{\lambda_0} < 0$ the effective coupling constant grows as a positive power of $1/\Lambda$. The borderline value $d_{\lambda_0} = 0$ corresponds to the case, when the small-scale divergences are logarithmic and therefore related to the large-scale logarithmic divergences, which can be dealt with by the standard methods of the quantum field theory. Therefore, the analysis of the large-momentum divergences of the model has to be carried out at the critical value $b_c = 1$ instead of $b = 2$, which corresponds to logarithmic divergences in the 1PI vertex graphs of the model.

The bare propagator g_0 is retarded, therefore all graphs with closed loops of more than one g_0 vanish (these are the only graphs, which formally contain integrals over the Laplace variable and longitudinal momenta). The closed loop with one bare propagator, which corresponds to the integral $\int \mathrm{d}s\, \mathrm{d}k\, \mathrm{d}p\, k g_0(s, k, p)$, vanishes, since the integrand is an odd function of the longitudinal momentum k. In particular, there are no graphs corresponding to 1PI Green functions with ψ-legs only, therefore $N_\varphi \geqslant 1$ always. One could worry about the coefficient $d - 3$ at N_φ, which, at least formally, could lead to an infinite number of types of superficially divergent graphs (i.e. graphs with $\delta' \geqslant 0$) below three dimensions. However, it is not difficult to see that, as in the short-range case [7], there is only one superficially divergent graph in the model: the 1-loop self-energy graph. Let us denote by $I_{\varphi\tilde{\varphi}}$ the number of g_0-propagators in an arbitrary 1PI graph of the model, then from $V = I_{\varphi\tilde{\varphi}} + N_\varphi$ we obtain the relation $V \geqslant N_\varphi + 1$ for all 1PI graphs, where the equality is achieved *only for the 1-loop self-energy graph* since it is the only 1PI graph, which contains exactly one g_0-propagator. Substituting this inequality in the relation (6), we find that for all graphs the actual degree of divergence has the upper bound $\delta' \leqslant d+3-b-(d-1-b)N_\varphi - bN_\psi$. At the critical value $b = 1$ and for $N_\varphi \geqslant 1$ we see that $\delta' = 0$ for the 1-loop self-energy graph, and $\delta' < 0$ for all other 1PI graphs, which, consequently, are superficially convergent. Here we have assumed that $d \geqslant 2$, which is a natural condition, since there is not much sense in dividing the position vector in longitudinal and transverse components below two dimensions.

Hence, in the minimal subtraction scheme there is only one renormalization constant Z, and the renormalized action may be written in the form

$$S_{\mathrm{R}} = -\frac{1}{2\lambda\mu^\varepsilon}\int \mathrm{d}y\,\mathrm{d}y'\,\psi C_1^{-1}\psi + \int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\tilde{\varphi}\left[\frac{\partial}{\partial t} - D^{\mathrm{T}}\frac{\partial^2}{\partial y^2} - ZD^{\mathrm{L}}\frac{\partial^2}{\partial x^2} - \psi(y)\frac{\partial}{\partial x}\right]\varphi \tag{7}$$

where we have introduced the renormalized diffusion constants D^{L} and D^{T}, renormalized coupling constant λ, and a scale-setting parameter μ of dimension of transverse momentum in order to make the renormalized coupling constant dimensionless under the scale transformation (5). The parameter ε is defined as $\varepsilon = 2(1-b)$. In general, the parameters λ and D differ from their bare counterparts at most by a finite renormalization factor, but in the minimal subtraction scheme $\lambda\mu^\varepsilon = \lambda_0$ and $D = D_0$, whereas $D_0^{\mathrm{L}} = ZD^{\mathrm{L}}$. In practical calculations, it is convenient to use the *basic action* S_{B}, the expression for which is obtained from (7) by setting $Z = 1$. The renormalized action is then the sum of the basic action and the *counterterms*, which arise in the course of the renormalization of the model to cancel the (ultra-violet) divergences.

The only superficially divergent 1PI graph yields the following contribution to the 'self-energy' function $\Sigma(k,p)$ (in terms of the basic action)

$$\begin{aligned}\Sigma_1(k,0) &= -k^2\lambda\mu^\varepsilon\int\frac{\mathrm{d}q}{(2\pi)^{d-1}}\frac{1}{(s+D^{\mathrm{L}}k^2+D^{\mathrm{T}}q^2)(q^2)^{(d-1)/2-b}} \\ &= -\frac{k^2\lambda\mu^\varepsilon\Gamma(1-\varepsilon/2)\Gamma(\varepsilon/2)}{(4\pi)^{(d-1)/2}(D^{\mathrm{T}})^{1-\varepsilon/2}\Gamma((d-1)/2)(s+D^{\mathrm{L}}k^2)^{\varepsilon/2}}\end{aligned} \tag{8}$$

where we have set the external transverse momentum equal to zero. Due to the super-renormalizability of the model, the one-loop expressions for the renormalization constant Z, the anomalous dimension γ of the longitudinal diffusion coefficient and the beta function extracted from (8), are perturbatively *exact* in the minimal subtraction scheme, which is used here. It should be noted that the asymptotic behaviour of the model is not determined by the coupling constant λ, but by a totally dimensionless expansion parameter u, which is dimensionless with respect to time, longitudinal and transverse coordinates, separately. From an inspection of the perturbation expansion, we infer the following expression for the parameter u

$$u = \lambda / D^{\mathrm{T}} D^{\mathrm{L}}.$$

In terms of this parameter, we obtain from (8)

$$Z = 1 - \frac{K_{d-1} u}{\varepsilon} \tag{9}$$

where we have introduced the quantity

$$K_d \equiv \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}.$$

The renormalized parameter D^{L} becomes scale-dependent, which is described by the quantity

$$\gamma(u) \equiv \mu \frac{\partial \ln D^{\mathrm{L}}}{\partial \mu}\bigg|_0 = -\mu \frac{\partial \ln Z}{\partial \mu}\bigg|_0 = -K_{d-1} u \tag{10}$$

called the anomalous dimension of the parameter D^{L}, i.e. the anomalous dimension of the longitudinal diffusion coefficient. In the definition (10) the subscript denotes that the derivative is taken at fixed values of the bare parameters D_0^{T}, D_0^{L} and λ_0. The asymptotic behaviour of the model is determined by the beta function, which is of the form

$$\beta(u) \equiv \mu \frac{\partial u}{\partial \mu}\bigg|_0 = u[-\varepsilon - \gamma(u)] = u(-\varepsilon + K_{d-1} u). \tag{11}$$

Dimensional analysis yields for the renormalized Green function G the relation

$$G(t, x, y; \mu, D^{\mathrm{L}}, D^{\mathrm{T}}, u) = \frac{R(x/\sqrt{D^{\mathrm{L}} t}, y\mu, t\mu^2 D^{\mathrm{T}}, u)}{(D^{\mathrm{L}})^{1/2} (D^{\mathrm{T}})^{(d-1)/2} t^{d/2}}. \tag{12}$$

Together with the basic RG equation

$$\left[\mu \frac{\partial}{\partial \mu} + \gamma(u) D^{\mathrm{L}} \frac{\partial}{\partial D^{\mathrm{L}}} + \beta(u) \frac{\partial}{\partial u}\right] G = 0$$

which expresses the independence of the Green function G_0 of the arbitrary scaling parameter μ, the relation (12) leads to the equation

$$\left[2t \frac{\partial}{\partial t} + \left(1 - \frac{\gamma(u)}{2}\right) x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \beta(u) \frac{\partial}{\partial u} + d - \frac{\gamma(u)}{2}\right] G = 0 \tag{13}$$

from which the asymptotic behaviour of the renormalized Green function G may be inferred.

If the running coupling constant $\bar{u}$ is considered as a function of time, then the exact expressions (10) and (11) yield the solution of equation (13) in a closed form:

$$G(t,x,y;\mu,D^{\mathrm{L}},D^{\mathrm{T}},u)=\frac{G(1,\bar{x},yt^{-1/2};\mu,D^{\mathrm{L}},D^{\mathrm{T}},\bar{u})}{t^{d/2+\varepsilon/4}[K_{d-1}u/\varepsilon+(1-K_{d-1}u/\varepsilon)t^{-\varepsilon/2}]^{1/2}} \tag{14}$$

where $\bar{x}$ and $\bar{u}$ are the first integrals of the equation (13):

$$\bar{x}=\left(\frac{x}{t^{1/2+\varepsilon/4}}\right)\sqrt{\frac{\bar{u}}{u}} \tag{15}$$

and

$$\bar{u}=\left(\frac{\varepsilon}{K_{d-1}}\right)\frac{1}{1-(1-\varepsilon/K_{d-1}u)t^{-\varepsilon/2}}. \tag{16}$$

The beta function (11) has two zeros (fixed points of the RG): the Gaussian fixed point $u^*_{\mathrm{G}}=0$ and the non-trivial fixed point $u^*=\varepsilon/K_{d-1}$, of which the former is infrared stable for $1<b<(d-1)/2$, and the latter for $b<1$, as may be seen from (16), where $\bar{u}\to 0$ in the limit $t\to\infty$, when $\varepsilon=2(1-b)<0$, whereas $\bar{u}\to\varepsilon/K_{d-1}$, when $\varepsilon>0$. Therefore, the asymptotic behaviour for correlations which decay rapidly enough corresponds to the usual diffusion, whereas for slowly falling-off correlations the anomalous behaviour governed by the non-trivial fixed point u^* occurs. Corrections to the usual diffusion result at $b=1$ logarithmic.

From the relations (14)–(16) it follows that in the transverse directions the diffusion, as described by the the long-time behaviour of the mean-square displacement, is normal, whereas for $0<b<1$ the anomalous dimension of the longitudinal diffusion coefficient has a non-trivial value: $\gamma\equiv\gamma(u^*)=-\varepsilon$, which, in particular, implies that in the longitudinal direction the behaviour of the mean-square displacement is superdiffusive: by definition $\langle\overline{x^2(t)}\rangle=\int\mathrm{d}x\,\mathrm{d}y\,G(t,x,y;\mu,D^{\mathrm{L}},D^{\mathrm{T}},u)$, and using the equations (14)–(16) we arrive at the relation

$$\langle\overline{x^2(t)}\rangle\sim t^{1+\varepsilon/2}=t^{2-b} \tag{17}$$

in which the value of the power $2-b$ is perturbatively exact in the sense that the higher order terms of the ε-expansion of the power of t vanish identically. The relation (17) determines the value of the exponent ν, defined by $\langle x^2(t)\rangle\sim t^{2\nu}$, as $\nu=1-b/2$. At $b=1$ the mean-square displacement grows as

$$\langle\overline{x^2(t)}\rangle\sim t\ln t.$$

At the limit $b\to(d-1)/2$, the value of the exponent ν coincides with that for the case of short-range correlations [7]. This is not at all a trivial fact in the isotropic case [3], and here it occurs due to the super-renormalizability of the field theory.

3. Diffusion in a unidirectional random field with longitudinal long-range correlations

In the case of diffusion in a random unidirectional velocity field with long-range correlations along the direction of the field flow, the overall setup of the problem is the same as in the previous section. However, the random field ψ now also depends on the longitudinal coordinate, and this has the important physical consequence that the effective velocity field $F = e_x\psi(x, y)$ ceases to be divergenceless. The correlation function has the asymptotic behaviour

$$\langle\psi(x,y)\psi(x',y')\rangle \sim \frac{\delta(y-y')}{|x-x'|^{2a}} \qquad |x-x'| \to \infty$$

and is assumed to be locally integrable with respect to the longitudinal coordinate $x - x'$. Arguments similar to those of the preceding section allow the original correlation function to be replaced by an effective one of the form

$$\langle\psi(x,y)\psi(x',y')\rangle = \frac{(1-2a)\lambda_0\delta(y-y')}{|x-x'|^{2a}} \equiv \lambda_0 C_2(x-x', y-y') \qquad 0 < a < \tfrac{1}{2}$$

$$\langle\psi(x,y)\psi(x',y')\rangle = 2\lambda_0\delta(x-x')\delta(y-y') \equiv \lambda_0 C_2(x-x', y-y') \qquad a \geqslant \tfrac{1}{2}.$$

Here, the long-range correlation function has been chosen such that in the limit $a \to 0$ we recover the correlation function of the original problem with unidirectional convection [7]. Also, due to the factor 2 in the short-range correlation function, the function C_2 is a continuous function of the parameter a in the limit $a \to \frac{1}{2}$.

In the long-range case we obtain for the dimensions of the bare coupling constant λ_0 the values $d^{\mathrm{s}}_{\lambda_0} = 2$, $d^{\mathrm{L}}_{\lambda_0} = -2 - 2a$, and $d^{\mathrm{T}}_{\lambda_0} = 1 - d$, and therefore $d_{\lambda_0} = 3 - 2a - d$. In this case the model has the usual critical dimension $d_c = 3 - 2a$, and we choose $\epsilon = 3 - 2a - d$. The random field ψ is now a function of all the spatial coordinates, therefore the longitudinal momentum integrals are also present. The actual degree of divergence δ' now takes into account longitudinal momenta both in the propagators and in the vertices (the latter factorize at the external legs), thus

$$\delta' = d - \tfrac{1}{2}V(3-2a-d) - (d-1)N_\varphi - \tfrac{1}{2}(d-1+2a)N_\psi. \qquad (18)$$

From here it follows that the model is renormalizable but not super-renormalizable: it has an infinite set of superficially divergent graphs, since at the critical dimension $d = 3 - 2a$ the number of vertices V disappears from the relation (18). Moreover, apart from the superficially linearly divergent self-energy graphs, the vertex graphs corresponding to the 1PI Green function $\Gamma_{\varphi\tilde{\varphi}\psi}$ also possess logarithmic divergences, and the renormalization of the model requires two renormalization constants Z and Z_1, which enter the renormalized action in the following way

$$S_{\mathrm{R}} = -\frac{1}{2\lambda\mu^\epsilon}\int \mathrm{d}y\,\mathrm{d}y'\,\mathrm{d}x\,\mathrm{d}x'\,\psi C_2{}^{-1}\psi + \int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\tilde{\varphi}\left[\frac{\partial}{\partial t} - D^{\mathrm{T}}\frac{\partial^2}{\partial y^2} - Z D^{\mathrm{L}}\frac{\partial^2}{\partial x^2}\right]\varphi$$
$$+ Z_1\int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\varphi(t,x,y)\psi(x,y)\frac{\partial}{\partial x}\tilde{\varphi}(t,x,y)$$

where we have introduced the renormalized parameters in the same fashion as in the relation (7).

We choose the bare contribution to the 1PI vertex function $\Gamma_{\varphi(k,p)\tilde{\varphi}(-k,q)\psi(-p-q)}$ in the form $\Gamma_0(s,k,p,q) = \mathrm{i}k$, where s is the Laplace variable, and k is the longitudinal momentum flowing in the φ leg of the graph, and p and q are the transverse momenta flowing in its φ and $\tilde{\varphi}$ legs, respectively. In the limit of small momenta, the 1-loop contribution to the vertex function is then (in terms of the basic action)

$$\Gamma_1(s,k,0,0) = -\frac{\mathrm{i}k\lambda\mu^\varepsilon\Gamma(3/2-a)}{2^{d-2+2a}\pi^{d-1/2}\Gamma(a)} \times \left[\int \mathrm{d}l\,\mathrm{d}\boldsymbol{q}\,\frac{1}{(l^2)^{-1/2-a}(s+D^{\mathrm{L}}l^2+D^{\mathrm{T}}\boldsymbol{q}^2)^2} + \mathrm{O}(k^2)\right]$$
$$= -\frac{\mathrm{i}k\lambda\mu^\varepsilon a\Gamma(3/2-a)\Gamma((3-d-2a)/2)}{2^{d-2+2a}\pi^{d/2}(D^{\mathrm{T}})^{(d-1)/2}(D^{\mathrm{L}})^{1+a}s^{(3-d-2a)/2}} + \mathrm{O}(k^3)$$

from which we obtain

$$Z_1 = 1 + \frac{a\Gamma(3/2-a)u}{\pi^{3/2-a}\varepsilon} + \cdots$$

and

$$\gamma_1(u) = \frac{a\Gamma(3/2-a)u}{\pi^{3/2-a}} + \cdots. \tag{19}$$

The 1-loop contribution to the self-energy is

$$\Sigma_1(k,0) = -\frac{\lambda\mu^\varepsilon\Gamma(3/2-a)}{2^{d-2+2a}\pi^{d-1/2}\Gamma(a)}\int \mathrm{d}l\,\mathrm{d}\boldsymbol{q}\,\frac{k(k-l)}{(l^2)^{1/2-a}[s+D^{\mathrm{L}}(k-l)^2+D^{\mathrm{T}}\boldsymbol{q}^2]^2}$$
$$= -\frac{(1-2a)k^2\lambda\mu^\varepsilon\Gamma(3/2-a)\Gamma((3-d-2a)/2)}{2^{d-2+2a}\pi^{d/2}(D^{\mathrm{T}})^{(d-1)/2}(D^{\mathrm{L}})^{a}s^{(3-d-2a)/2}} + \mathrm{O}(k^3)$$

from which

$$Z = 1 - \frac{(1-2a)\Gamma(3/2-a)u}{\pi^{3/2-a}\varepsilon} + \cdots$$

and

$$\gamma(u) = -\frac{(1-2a)\Gamma(3/2-a)u}{\pi^{3/2-a}} + \cdots. \tag{20}$$

The beta function now contains both γs

$$\beta = u[-\varepsilon + 2\gamma_1(u) - (1+a)\gamma(u)] \tag{21}$$

and the fixed-point equation no longer fixes the anomalous dimension $\gamma = \gamma(u^*)$ to all orders in ε. From (19), (20) and (21) we obtain to the order $\mathrm{O}(u^2)$

$$\beta = u\left[-\varepsilon + \frac{(1-a)(1+2a)\Gamma(3/2-a)u}{\pi^{3/2-a}}\right]$$

from which it follows that below the critical dimension $d < d_c = 3 - 2a$ there is an infrared stable fixed point of the RG: $u^* = \pi^{3/2-a}\varepsilon/(1-a)(1+2a)\Gamma(3/2-a)$, at which the anomalous dimension of the longitudinal diffusion coefficient is equal to

$$\gamma = -\frac{(1-2a)\varepsilon}{(1-a)(1+2a)} + O(\varepsilon^2)$$

leading to superdiffusive behaviour, and at the critical dimension logarithmic corrections to normal diffusion take place

$$\langle \overline{x^2(t)} \rangle \sim t^{1+(1-2a)\varepsilon/2(1-a)(1+2a)} \qquad d < 3 - 2a \qquad 0 < a < \tfrac{1}{2} \tag{22}$$

$$\langle \overline{x^2(t)} \rangle \sim t(\ln t)^{(1-2a)/(1-a)(1+2a)} \qquad d = 3 - 2a \qquad 0 < a < \tfrac{1}{2} \tag{23}$$

whereas above the critical dimension the diffusion is normal. Expressions (22) and (23) assume the values of the short-range unidirectional convection problem [7] in the limit $a \to 0$, and the anomalies vanish in the other interesting limit $a \to 1/2$. In the transverse directions the diffusion is normal regardless of the values of a, b and d.

4. Diffusion in a random field with long-range longitudinal and transverse correlations

The problem of diffusion in a unidirectional random velocity field with long-range correlations in both longitudinal and transverse directions is described by the random field $\psi(x, y)$ with the pair correlation function locally integrable with respect to longitudinal and transverse coordinates separately and with the asymptotic behaviour

$$\langle \psi(x,y)\psi(x',y')\rangle \sim \frac{1}{|x-x'|^{2a}|y-y'|^{2b}} \qquad |x-x'| \to \infty \qquad |y-y'| \to \infty .$$

For $a < \frac{1}{2}$ and $b < (d-1)/2$ we replace, by analogy with the preceding treatment, the original correlation function by an effective one of the form

$$\begin{aligned}\langle \psi(x,y)\psi(x',y')\rangle &= \lambda_0 C_3(x-x', y-y') \\ &\equiv \frac{2^{2b}\Gamma(b)(1-2a)\lambda_0}{(4\pi)^{(d-1)/2}\Gamma((d-1)/2-b)|x-x'|^{2a}|y-y'|^{2b}} \\ &\quad 0 < a < \tfrac{1}{2} \qquad 0 < b < (d-1)/2 .\end{aligned}$$

We only consider here this correlation function, since the cases corresponding to other possible values of the exponents a and b reduce to the problems treated in the preceding sections.

The dimensions of the bare coupling constant are $d^{s}_{\lambda_0} = 2$, $d^{L}_{\lambda_0} = -2 - 2a$, and $d^{T}_{\lambda_0} = -2b$, therefore $d_{\lambda_0} = 2(1 - a - b)$. Thus, there is a critical line $a + b = 1$ in the space of the parameters a and b, but no critical dimension of the position space in this case. We define $\varepsilon = 2(1 - a - b)$. Power counting yields for the actual degree of divergence the expression

$$\delta' = d - (1 - a - b)V - (d-1)N_{\varphi} - (a+b)N_{\psi}$$

from which it follows that in the generic case there is a superficially linearly divergent two-point function $\Gamma_{\varphi\tilde{\varphi}}$ and a logarithmically divergent three-point function $\Gamma_{\varphi\tilde{\varphi}\psi}$ in the model. At two dimensions the four-point function $\Gamma_{\varphi\tilde{\varphi}\varphi\tilde{\varphi}}$ also has a degree of divergence equal to zero indicating a possible logarithmic divergence. However, at two dimensions at least one of the conditions $a < \frac{1}{2}$ and $b < (d-1)/2$ cannot be fulfilled, consequently, in the effective correlation function the corresponding power function has to be replaced by a delta function, and we return to one of the previous cases, in which there are no difficulties with the higher order Green functions. Consequently, there are only two effectively logarithmically divergent Green functions in the model, and two renormalization constants suffice to make the model finite. The renormalized action is

$$S_{\mathrm{R}} = -\frac{1}{2\lambda\mu^{\epsilon}}\int \mathrm{d}y\,\mathrm{d}y'\,\mathrm{d}x\,\mathrm{d}x'\;\psi C_3{}^{-1}\psi + \int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\;\tilde{\varphi}\left[\frac{\partial}{\partial t} - D^{\mathrm{T}}\frac{\partial^2}{\partial y^2} - Z D^{\mathrm{L}}\frac{\partial^2}{\partial x^2}\right]\varphi$$
$$+ Z_1\int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\;\varphi(t,x,y)\psi(x,y)\frac{\partial}{\partial x}\tilde{\varphi}(t,x,y).$$

Choosing the bare part of the 1PI vertex function $\Gamma_{\varphi(k,p)\tilde{\varphi}(-k,q)\psi(-p-q)}$ in the form $\Gamma_0(s,k,p,q) = \mathrm{i}k$, we obtain the 1-loop contribution to the vertex function at small momenta (in terms of the basic action) in the form

$$\Gamma_1(s,k,0,0) = -\frac{\mathrm{i}k\lambda\mu^{\epsilon}\Gamma(3/2-a)}{2^{d-2+2a}\pi^{d-1/2}\Gamma(a)}$$
$$\times\left[\int \mathrm{d}l\,\mathrm{d}q\,\frac{1}{(l^2)^{-1/2-a}(q^2)^{(d-1)/2-b}(s+D^{\mathrm{L}}l^2+D^{\mathrm{T}}q^2)^2} + \mathrm{O}(k^2)\right]$$
$$= -\frac{\mathrm{i}k\lambda\mu^{\epsilon}a\Gamma(b)\Gamma(3/2-a)\Gamma(1-a-b)}{2^{d-2+2a}\pi^{d/2}\Gamma((d-1)/2)(D^{\mathrm{T}})^{b}(D^{\mathrm{L}})^{1+a}s^{1-a-b}} + \mathrm{O}(k^3) \qquad (24)$$

and the contribution to the self-energy in the form

$$\Sigma_1(k,0) = -\frac{\lambda\mu^{\epsilon}\Gamma(3/2-a)}{2^{d-2+2a}\pi^{d-1/2}\Gamma(a)}$$
$$\times\int \mathrm{d}l\,\mathrm{d}q\,\frac{k(k-l)}{(l^2)^{1/2-a}(q^2)^{(d-1)/2-b}[s+D^{\mathrm{L}}(k-l)^2+D^{\mathrm{T}}q^2]^2}$$
$$= -\frac{(1-2a)k^2\lambda\mu^{\epsilon}\Gamma(b)\Gamma(3/2-a)\Gamma(1-a-b)}{2^{d-2+2a}\pi^{d/2}\Gamma((d-1)/2)(D^{\mathrm{T}})^{b}(D^{\mathrm{L}})^{a}s^{1-a-b}} + \mathrm{O}(k^3). \qquad (25)$$

In the relations (24) and (25) there are two natural choices of the totally dimensionless expansion parameter: $\lambda(D^{\mathrm{T}})^{-b}(D^{\mathrm{L}})^{b-2}$, which corresponds to the parameter μ of the dimension of the longitudinal momentum, and $\lambda(D^{\mathrm{T}})^{a-1}(D^{\mathrm{L}})^{-1-a}$, which corresponds to μ of the dimension of the transverse momentum. We choose the latter possibility and define

$$u \equiv \frac{\lambda}{(D^{\mathrm{T}})^{1-a}(D^{\mathrm{L}})^{1+a}}.$$

In terms of this parameter, the renormalization constants are

$$Z_1 = 1 + \frac{aEu}{\varepsilon} + \cdots \qquad Z = 1 - \frac{(1-2a)Eu}{\varepsilon} + \cdots \tag{26}$$

where we have defined

$$E \equiv \frac{\Gamma(1-a)\Gamma(3/2-a)}{2^{d+2a-3}\pi^{d/2}\Gamma((d-1)/2)}.$$

Equations (26) yield

$$\gamma_1(u) = aEu + \cdots \qquad \gamma(u) = -(1-2a)Eu + \cdots. \tag{27}$$

The beta function from (26) and (27) is of the form

$$\beta = u[-\varepsilon + 2\gamma_1(u) - (1+a)\gamma(u)] = u[-\varepsilon + (1-a)(1+2a)Eu + \mathrm{O}(u^2)]$$

from which it follows that for $a + b < 1$ there is an infrared stable fixed point of the RG $u^* = \varepsilon/(1-a)(1+2a)E$, at which the anomalous dimension of the longitudinal diffusion coefficient is formally the same as in the preceding section (the parameter $\varepsilon = 2(1-a-b)$ is different from that in the previous sections)

$$\gamma = -\frac{(1-2a)\varepsilon}{(1-a)(1+2a)} + \mathrm{O}(\varepsilon^2)$$

leading to superdiffusive behaviour, and at the critical line $a + b = 1$ logarithmic corrections to normal diffusion occur:

$$\langle\overline{x^2(t)}\rangle \sim t^{1+(1-2a)\varepsilon/2(1-a)(1+2a)} \qquad a+b<1 \qquad 0<a<\tfrac{1}{2}$$
$$0<b<(d-1)/2$$
$$\langle\overline{x^2(t)}\rangle \sim t(\ln t)^{(1-2a)/(1-a)(1+2a)} \qquad a+b=1 \qquad 0<a<\tfrac{1}{2}$$
$$0<b<(d-1)/2$$

whereas for $a + b > 1$, $a < \frac{1}{2}$, and $b < (d-1)/2$ the diffusion is normal, as is also the case in the transverse directions for all values of a and b.

5. Conclusion

In this paper a renormalization group analysis has been carried out of three generalizations of the recently proposed model of diffusion in a random unidirectional velocity field [5, 6]. The cases of: (i) a random field independent of the coordinate along the velocity direction and long-range correlations in the transverse directions; (ii) a random field with long-range correlations in the direction of the velocity and short-range correlations in the transverse directions; and (iii) a random field with long-range correlations in both the longitudinal and transverse directions, have been considered. The critical values of decay exponents of the correlation function and the critical dimension have been determined, and the anomalous dimension of the longitudinal diffusion coefficient has been calculated in the leading order of the ε expansion. In the case of a random field independent of the coordinate in the velocity direction the results are perturbatively exact.

In all cases the anomalous behaviour is superdiffusive with the following long-time asymptotics of the mean-square displacement of a tracer particle in the longitudinal direction:

(i) for $0 < b < (d-1)/2$, $\overline{\langle x^2(t)\rangle} \sim t^{2-b}$, if $b < 1$ and $\overline{\langle x^2(t)\rangle} \sim t \ln t$, when $b = 1$; these results are exact in the ε-expansion.

(ii) For $0 < a < \frac{1}{2}$, $\overline{\langle x^2(t)\rangle} \sim t^{1+(1-2a)\varepsilon/2(1-a)(1+2a)}$, when $d < 3 - 2a$ and $\overline{\langle x^2(t)\rangle} \sim t(\ln t)^{(1-2a)/(1-a)(1+2a)}$, if $d = 3 - 2a$.

(iii) For $0 < a < \frac{1}{2}$, $0 < b < (d-1)/2$, $\overline{\langle x^2(t)\rangle} \sim t^{1+(1-2a)\varepsilon/2(1-a)(1+2a)}$, when $a + b < 1$ and $\overline{\langle x^2(t)\rangle} \sim t(\ln t)^{(1-2a)/(1-a)(1+2a)}$, if $a + b = 1$.

For other values of the parameters a, b and d the diffusion in the biased direction is normal: $\overline{\langle x^2(t)\rangle} \sim t$. In the transverse directions diffusion is normal regardless of the values of a, b and d.

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